1. Write the Navier-Stokes equation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = \nu\nabla^2\mathbf{u}$$

in terms of dimensionless variables and obtain an expression for the Reynolds number.

Estimate the magnitude of the Reynolds number for:

(a) flow past the wing of a jumbo jet,
(b) a human swimmer,
(c) a thick layer of treacle draining off a spoon,
(d) a bacterium swimming in water.

Take the kinematic viscosity ν to be $10^{-6}\mathrm{m}^2\mathrm{s}^{-1}$ for water, $1.5\times10^{-5}\mathrm{m}^2\mathrm{s}^{-1}$ for air and $10^{-1}\mathrm{m}^2\mathrm{s}^{-1}$ for treacle.

2. For a nematic liquid crystal the Landau-de Gennes expansion of the free energy (ignoring elastic terms) can be written

$$F = F_0 + aQ_{\alpha\beta}Q_{\beta\alpha} - bQ_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\alpha} + cQ_{\alpha\beta}Q_{\beta\gamma}Q_{\gamma\delta}Q_{\delta\alpha}.$$

(a) For a uniaxial, uniformly ordered nematic, with order of magnitude q, write down the form of the Q-tensor, and show that the free energy reduces to

$$F = F_0 + \frac{3}{2}aq^2 - \tilde{b}q^3 + \tilde{c}q^4.$$

(b) Plot $F - F_0$ as a function of q and show that there is a first order transition from a disordered to an ordered state at $a = \tilde{b}/6\tilde{c}$, $q = \tilde{b}/2\tilde{c}$.

(c) How does this change in 2D?

3. The velocity field produced by a $+1/2$ defect can be approximated by

$$v_+^a(r,\phi) = \frac{\zeta}{12\eta}\left\{(3(R-r) + r\cos 2\phi)\hat{\boldsymbol{x}} + r\sin 2\phi\ \hat{\boldsymbol{y}}\right\}. \tag{1}$$

For a $-1/2$ defect

$$v_-^a(r,\phi) = \frac{\zeta r}{12\eta R}\{((3r/4 - R)\cos 2\phi - R\cos 4\phi/5)\hat{\boldsymbol{x}} + ((3r/4 - R)\sin 2\phi + R\sin 4\phi/5)\hat{\boldsymbol{y}}\}. \tag{2}$$

Use these expressions to plot the flow field around a $+1/2$ and a $-1/2$ active defect. (See Appendix of Giomi et al Philosophical Transactions: Mathematical, Physical and Engineering Sciences, 372:0365, 2014 for details of how the formulas are derived.)

Trickier problem

4. A simple model of a self-propelled particle i which gives the correct far flow field is a rod centred at $\boldsymbol{x}_i$ along $\hat{\boldsymbol{n}}_i$ with equal and opposite forces $\pm f\hat{\boldsymbol{n}}_i$ acting at its ends, at $\boldsymbol{x}_i + a\hat{\boldsymbol{n}}_i$ and $\boldsymbol{x}_i - a'\hat{\boldsymbol{n}}_i$.

(i) Write down the force per unit volume due to a collection of these swimmers.

(ii) Taylor expand, and hence identify the stress.

(iii) Coarse-grain to shown that the active stress is $-\zeta Q$.

If you get stuck see R. A. Simha and S. Ramaswamy, Phys. Rev. Lett. 89, 058101 (2002).

Open ended problem

5. Active topological defects

The aim of this problem is to engage with the literature on active nematics. There are now many examples of active (and passive) topological defects in biological materials. Can you find some of these and explain
- why the material is behaving as an active nematic (or maybe not)
- does the dynamics fit standard active nematic theories, or do we have to worry about differences?
- do the topological defects have any biological relevance?

One way to find relevant papers is a forward search on Saw et al, Nature 544, 212 (2017)
https://www.nature.com/articles/nature21718